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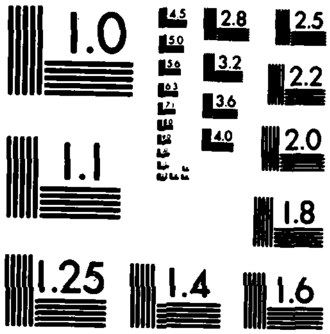
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ELASTIC FIELDS UNDER A SPHERICAL INDENTER

FINAL TECHNICAL REPORT

By

E.H. Yoffe

March 1984

UNITED STATES ARMY
EUROPEAN RESEARCH OFFICE OF THE U.S. ARMY
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Modified Hertz theory for spherical indentation

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Abstract

The elastic field in a half-space indented by a rigid sphere is accurately described by Hertz' equations if the contact radius is small compared with that of the sphere. For wider contact the equations are not expected to be reliable, but are nevertheless often used for convenience. This paper describes some of the errors which may develop as contact widens, and presents a second elastic field which provides a first order correction. From this it is possible to estimate the probable error introduced in a particular variable by applying the Hertz equations beyond the recommended range.

Some useful relations between some familiar elastic indentation fields are also reported.

1. Introduction

It was clearly stated by Hertz (1881) that his solution for the elastic field due to pressure between spheres should be applied only to contact regions of radius small compared with that of the spheres. His formulae have, for many years, been applied to the indenting of a flat elastic specimen by a hard sphere, and found accurate and reliable in the range $a/R < 0.1$, where a, R are the radii of contact and of the sphere respectively.

The theory has also been used beyond this value, and the Hertz formulae now form the basis of an ever-widening range of methods for testing and determining the properties of materials. Because of this, it is of interest to calculate another term in the elastic solution, extending its range to larger a/R, in order to make clear the conditions under which this second term may safely be ignored.

In this paper we imagine a rigid sphere pressed on an elastic half-space, and calculate the displaced form of the initially flat surface using Hertz' equations. It is shown that there is some misfit between the deformed surface and the sphere, and a second elastic field is developed as a correction. The theory of this second stress field is described in detail in §3, and some relevant comments of a more general nature are made in §4. The latter section may, however, be omitted by the reader who wishes to proceed to the applications in §5. Finally the method of correction is demonstrated in the practical case of determining the elastic modulus E of a pliant material.

The displacements, stresses and strains are those of first order linear elasticity theory throughout.

2. Errors in Hertz' solution for large a

For a rigid spherical indenter pressed with load F on the flat surface of an elastic half-space, the components of displacement u_ρ and w as given by the Hertz theory are:

$$u_\rho = \frac{-(1 - 2\nu)F}{4\pi G\rho}\left(1 - (1 - \rho^2/a^2)^{3/2}\right)$$

$$w = \frac{3(1 - \nu)F}{16\, G\, a}\left(2 - \rho^2/a^2\right) \qquad (1)$$

within the contact area $\rho < a$. Here ν and G are Poisson's ratio and the shear modulus respectively, and the cylindrical axes ρ, ϕ, Z are as indicated in fig.1.

Near the Z axis, these displacements give very close fit of the deformed surface to a sphere of radius R_H, where

$$1/R_H = 3(1 - \nu)F/8G\, a^3 \qquad (2)$$

since the w displacement defines a parabola with this value of maximum curvature. But as ρ increases the parabola diverges from the sphere, and the w displacement therefore introduces a growing gap, while the u_ρ component tends to close it. For large values of Poisson's ratio, ν, the component w has the greater effect, since $u_\rho \to 0$ as $\nu \to 0.5$. So for large ν the Hertz solution creates a gap between sphere and surface in the outer part of the contact region, despite the transmission of pressure there. In other words, contact is lost over part of the "contact area".

For small values of ν the component u_ρ is dominant, bringing the indented surface actually within the volume of the sphere. For intermediate values of ν there is an approximate balance between the two, gap or overlap depending on the extent of contact a/R.

In fig.2 are curves indicating the gaps and overlaps implicit in the calculated at $\rho = 0.9a$ in each case, since this is near the position of maximum u_ρ. The error for small ν is always an overlap, which increases with decreasing ν and with increasing a/R.

These errors, although small, show a logical inconsistency in applying the Hertz theory outside the range a/R .2, since the prescribed pressure cannot readily be transmitted across a gap, nor can sphere and specimen occupy the same point in space. The value of the contact radius, a, for a given load and sphere radius, becomes doubtful also, but it is not immediately apparent whether larger or smaller values would reduce the errors in fig.2. Since the values of a are commonly used in estimates of strength and toughness derived from the indentation of hard, brittle materials, the question of their accuracy becomes important.

To investigate this, better fitting contact may be made by combining the Hertz field with another exact elastic one (the parabolic pressure field) in proportions $1 + d : -d$, where d is a small parameter, so that each of the two separate solutions, and the combined one, applies the same load F to the same contact area $\rho < a$. This means that a is fixed, while the parameter d is varied and adjusted so as to make the displaced surface as nearly spherical as possible over the whole contact area. The value of d is found in this way for each value of F, and thence a modified form of equation (2).

But first the new elastic field must be described.

3. The parabolic distribution of pressure

A force F applied normally to the contact area may be distributed as pressure p given by:

$$p = 2F(a^2 - \rho^2)/\pi a^4 \quad \text{for } \rho < a$$
$$= 0 \quad \text{for } \rho > a \qquad (3)$$

which resembles a paraboloid of revolution, with value $2F/\pi a^2$ on the z axis, zero at the contact edge. This case was mentioned by Boussinesq (1885 p149) who calculated the normal displacement w of the compressed surface as

$$\begin{aligned} w &= \frac{\lambda + 2\mu}{\pi^2 \mu(\lambda+\mu)} \cdot \frac{2\pi}{3a} \quad \text{at } \rho = 0 \\ &= \frac{\lambda + 2\mu}{\pi^2 \mu(\lambda+\mu)} \cdot \frac{8}{9a} \quad \text{at } \rho = a \end{aligned} \tag{4}$$

where his $(\lambda + 2\mu)/\mu(\lambda + \mu) = 2(1 - \nu)/G$. Further details of this field do not appear to have been published, but may readily be found by using Boussinesq' harmonic functions as described by Love (1929).

If a function χ is defined by the equation

$$\chi(\rho, z) = \iint p \log (z + r)\, dS \tag{5}$$

where dS is an element of the contact area $0 < \rho < a$, r its distance from the field point (ρ, ϕ, z), and if

$$\begin{aligned} V(\rho, z) &= \partial\chi/\partial z \\ &= \iint p\, r^{-1}\, dS \end{aligned} \tag{6}$$

then χ and V are harmonic functions from which we may calculate the stresses and displacements, throughout the region $z > 0$, caused by the given applied pressure p. The stresses are given by:

$$\begin{aligned} 2\pi\sigma_\rho &= 2\nu \frac{\partial V}{\partial z} - (1 - 2\nu) \frac{\partial^2 \chi}{\partial \rho^2} - z \frac{\partial^2 V}{\partial \rho^2} \\ 2\pi\sigma_\phi &= 2\nu \frac{\partial V}{\partial z} - \frac{(1 - 2\nu)}{\rho} \frac{\partial \chi}{\partial \rho} - \frac{z}{\rho} \frac{\partial V}{\partial \rho} \\ 2\pi\sigma_z &= \frac{\partial V}{\partial z} - z \frac{\partial^2 V}{\partial z^2} \\ 2\pi\tau_{\rho z} &= - z \frac{\partial^2 V}{\partial \rho \partial z} \end{aligned} \tag{7}$$

The other shear stresses are zero. For the displacements u_ρ and w we have

$$4\pi G \quad u_\rho = -(1-2\nu)\frac{\partial \chi}{\partial \rho} - z\frac{\partial V}{\partial \rho}$$

$$4\pi G \; w = 2(1-\nu)V - z\frac{\partial V}{\partial z} \qquad (8)$$

and from these equations the whole field may in principle be determined, by substituting a particular form such as (3) for the pressure p in (5).

But it is not always easy to calculate χ and V from (5) and (6) as explicit functions of ρ and z, although V has been determined for the Hertz pressure distribution, and $\frac{\partial V}{\partial z}$ found for the case of a uniform pressure (Love, 1929). The function V is also known for pressures due to smooth rigid indenters of flat or conical form (Love 1939, Green 1947) but attempts to solve the present case of parabolic pressure (3) have not yet been successful.

However, much can be done with the surface and axial values of the functions, which are readily determined. We have, on $\rho = 0$,

$$V(o,z) = 2\pi\int_o^a \frac{p\,(\rho_1)}{(z^2+\rho_1{}^2)^{\frac{1}{2}}}\,\rho_1\,d\rho_1$$

$$= \frac{4F}{3a^4}\left[2(z^2+a^2)^{3/2} - 2z^3 - 3a^2z\right]$$

$$\frac{\partial V}{\partial z} = \frac{4F}{a^4}\left[2z\,(z^2+a^2)^{\frac{1}{2}} - 2z^2 - a^2\right]$$

$$= \frac{-2}{\rho}\frac{\partial \chi}{\partial \rho}$$

$$= -2\frac{\partial^2 \chi}{\partial \rho^2} \qquad (9)$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{8F}{a^4}\left[\frac{z^2}{(z^2+a^2)^{\frac{1}{2}}} + (z^2+a^2)^{\frac{1}{2}} - 2z\right]$$

$$= \frac{-2}{\rho}\frac{\partial V}{\partial \rho}$$

$$= -2\frac{\partial^2 V}{\partial \rho^2}$$

since V and χ have axial symmetry and satisfy Laplace's equation. For the function χ we may integrate V with respect to z:

$$\chi(o,z) = \int V\, dz \tag{10}$$

$$= \frac{F}{3a^4}\Big[\, 2z\,(a^2+z^2)^{3/2} + 3a^2 z(a^2+z^2)^{\frac{1}{2}} - 2z^4$$

$$+ 3a^4 \log\,(z + (a^2+z^2)^{\frac{1}{2}}) - 6\,a^2z^2 + \text{const.}\Big]$$

Equations (7) to (10) determine the stress components and w on the z axis.

To find their values at other points we observe that V as given by (9), but regarded as a function of a complex variable z, is an analytic function of z in a domain which includes the origin. It is therefore possible to express V as an integral of the same function of $(z + i\rho\cos\theta)$ (see Whittaker and Watson, 1946, p399). For, if

$$V(0,z) = g(z) \quad \text{in (9)}$$

then

$$V(\rho,z) = \frac{1}{\pi}\int_0^{\pi} g(z + i\rho\cos\theta)\, d\theta \tag{11}$$

in a region surrounding the origin. Similar relations may be written down for $\frac{\partial V}{\partial z}$ and $\frac{\partial^2 V}{\partial z^2}$ since they also are harmonic, but this does not apply to the derivatives with respect to ρ.

Although these integrals may be cumbersome for an arbitrary point (ρ,z), they can readily be evaluated on the surface, where $z = 0$ and $\rho < a$:

$$V(\rho,o) = \frac{1}{\pi}\int_0^{\pi} \frac{4F}{3a^4}\Big(2(a^2 - \rho^2\cos^2\theta)^{3/2}$$

$$+ 2i\rho^3\cos^3\theta - 3ia^2\rho\cos\theta\Big)\, d\theta$$

$$= \frac{16F}{3\pi a}\int_0^{\pi/2} (1 - \frac{\rho^2}{a^2}\cos^2\theta)^{3/2}\, d\theta$$

(in agreement with Boussinesq p.150)

$$= \frac{16F}{9\pi a}\Big[\, 2(2-k^2)E - (1-k^2)K\Big] \tag{12}$$

where E, K are complete elliptic integrals of modulus $k = \rho/a$.* Similarly

$$\frac{\partial V}{\partial z}(\rho,o) = \frac{4F}{a^4}(\rho^2 - a^2)$$

$$\chi(\rho,o) = \frac{F\rho^2}{4a^4}(4a^2 - \rho^2) + \text{const.}$$

$$\therefore \quad \frac{\partial \chi}{\partial \rho} = \frac{F\rho}{a^4}(2a^2 - \rho^2) \qquad (13)$$

These equations, with (7) and (8), give the displacements and stresses on the contact area. At points (ρ, o) on the free surface outside the contact region the stress components and u_ρ take their usual values corresponding to the load F (see Way, 1940), and w is found from Boussinesq' integral (1885 p.150).

The results for the parabolic pressure distribution are summarised below.

(i) On the z axis, where $\rho = o$:

$$u_\rho = 0$$

$$w = \frac{F}{3\pi G a^4}\Big[4(1-\nu)(a^2+z^2)^{3/2} - 6z^2(a^2+z^2)^{\frac{1}{2}} - 3(1-2\nu)a^2 z + 2(1+2\nu)z^3\Big]$$

$$\sigma_\rho = \frac{2F}{\pi a^4}\Big[(3+2\nu)\,z(a^2+z^2)^{\frac{1}{2}} - (3+2\nu)z^2 - (1+2\nu)a^2/2 - a^2 z(a^2+z^2)^{-\frac{1}{2}}\Big] \qquad (14_i)$$

$$\sigma_\phi = \sigma_\rho$$

$$\sigma_z = \frac{2F}{\pi a^4}\Big[2z^2 - a^2 - 2z(a^2+z^2)^{\frac{1}{2}} + 2a^2 z(a^2+z^2)^{-\frac{1}{2}}\Big]$$

---

* E is not Young's modulus.

(ii) For $z = o$, $\rho < a$:

$$u_\rho = -(1 - 2\nu)\ F\ \rho\ (2a^2 - \rho^2)/4\pi G a^4$$

$$w = 8(1 - \nu)\ F\left[2(2 - k^2)\ E - (1 - k^2)K\right]/9\pi^2 Ga$$

E,K being elliptic integrals of modulus $k = \rho/a$.

$$\sigma_\rho = F\left((3 + 2\nu)\ \rho^2 - 2(1 + 2\nu)a^2\right)/2\pi a^4$$

$$\sigma_\phi = F\left((1 + 6\nu)\ \rho^2 - 2(1 + 2\nu)a^2\right)/2\pi a^4 \qquad (14_{ii})$$

$$\sigma_z = -2F\ (a^2 - \rho^2)/\pi a^4$$

(iii) For $\rho > a$, $z = o$, $\frac{\partial\chi}{\partial\rho} = \frac{F}{\rho}$:

$$u_\rho = -(1 - 2\nu)F/4\pi G\rho \qquad (14_{iii})$$

$$\sigma_\rho = (1 - 2\nu)F/\ 2\pi\rho^2 = -\ \sigma_\phi$$

$$\sigma_z = 0$$

as for any symmetric distribution of the load F (Way 1940) but w is here:

$$w = 8(1 - \nu)F\rho\left[(2 - 3k^2)(1 - k^2)K + 2(2k^2 - 1)E\right]/9\pi^2 Ga^2 k^2$$

where the elliptic integrals now have modulus $k = {}^a/\rho$.

From equation (14_{ii}) the curvature of the displaced surface at the axis is

$$1/R_p = 2(1 - \nu)F/\pi Ga^3$$

and the vertical displacement there is

$$w_o = 4(1 - \nu)\ F/3\pi Ga \qquad (15)$$

so that this field gives a slightly deeper and more pointed indent than the Hertz formulae. The maximum pressure, at the centre of the contact area, is

$$-\ \sigma_z = 2F/\pi a^2 \qquad (16)$$

or twice the mean pressure.

This completes our description of the parabolic pressure field, which is used as a correction to the Hertz solution in §5. But first, since other contact problems may require similar corrections, we briefly describe in §4 a few simple pressure fields and some useful relations between them. This section may, however, be omitted by the reader who wishes to proceed to §5.

4. Some simple pressure fields

In linear elasticity theory many problems are solved by combining separate known stress fields in suitable proportions. This present paper makes use of the Hertz solution and the parabolic pressure field, but there are three other well-known axially-symmetric cases, namely the flat punch (Boussinesq 1885 p.158), the conical punch (Love 1939) and the case of uniform pressure (Boussinesq p.140, Love 1929). To these we may add the linear or conical pressure field. Some interesting properties of these fields and some relations between them are worth mentioning here.

Arranged in two groups of three, the six cases are displayed in Table I, with diagrams of the pressure distribution and the shape of each indentation, and with the corresponding formulae alongside. Since these fields may be combined linearly it is often possible to build the shape of a required pressure distribution or given indentation from these known fields. The process is simple if the contact area is the same in each case.

Alternatively, a given pressure distribution may be represented approximately by applying uniform pressure (B_1 in Table I) to smaller contact areas of radii a/n, $2a/n$ ... $(n-1)a/n$, a, so making a stepwise approach to the required curve, becoming closer as n increases. Both B_2 and B_3 (in Table I) may be regarded as integrals of B_1 in this way, but it is not so easy to build a required indentation shape, such as a parabola, A_2, or cone, A_3, by superposing solutions A_1 for varying contact, because the non-zero displacements w outside the contact must be included in the calculation.

Now having seen the parabolic pressure as an integral of simple uniform ones with respect to a, we may reverse this process and regard the whole stress field B_1 as a derivative of B_2 or B_3, and the flat punch A_1 as a derivative of Hertz, as follows.

As described by Love (1929) the elastic stress field of a particular pressure distribution may be derived from a single harmonic function χ, by using eqns. (5) to (8) in §3. For the Hertz case,

$$\chi_H(\rho,z) = \frac{3F}{2\pi a^3}\int_o^a\int_o^{2\pi}(a^2 - \rho_1{}^2)^{\frac{1}{2}} \log (z + r)\, \rho_1\, d\phi\, d\rho_1$$

where r is the distance between the field point (ρ, o, z) and a point (ρ_1, ϕ, o) in the contact area. If we write this:

$$a^3\chi_H = \frac{3F}{2\pi}\int_o^a\int_o^{2\pi}(a^2 - \rho_1{}^2)^{\frac{1}{2}} \log (z + r)\rho_1\, d\phi\, d\rho_1$$

then $a^3\chi_H$ is a continuous function of a for any *fixed* point (ρ,z). We may therefore differentiate this function with respect to a, and so obtain another harmonic function of ρ,z;

$$\frac{\partial}{\partial a}(a^3\chi_H) = \frac{3Fa}{2\pi}\int_o^a\int_o^{2\pi}\frac{\log(z + r)}{(a^2 - \rho_1{}^2)^{\frac{1}{2}}}\,\rho_1\, d\phi\, d\rho_1$$

(since the pressure is zero at $\rho_1 = a$). But for a load F applied with a rigid flat circular punch (case A_1) the fundamental harmonic function is χ_F, where

$$\chi_F = \frac{F}{2\pi a}\int_o^a\int_o^{2\pi}\frac{\log\ (z + r)}{(a^2 - \rho_1{}^2)^{\frac{1}{2}}}\,\rho_1\, d\phi\, d\rho_1$$

$$= \frac{\partial(a^3\chi_H)}{\partial a} \Big/ 3a^2 \qquad (17)$$

This is the simple relation between the two elastic fields, and by interchanging the order of differentiation it is possible to regard a component of the flat punch field as a simple derivative of the corresponding Hertz component.

For example, the displacement u_ρ in the contact area, for the flat punch is given by:

$$u_\rho \text{ (punch)} = \frac{1}{3a^2} \frac{\partial}{\partial a} (a^3 u_\rho \text{ (Hertz))}$$

$$= \frac{-(1 - 2\nu)F}{3a^2 \; 4\pi G\rho} \frac{\partial}{\partial a} (a^3 - (a^2 - \rho^2)^{3/2})$$

$$= \frac{-(1 - 2\nu)F}{4\pi G\rho a} (a - (a^2 - \rho^2)^{1/2})$$

Similarly the function V, that is $\frac{\partial \chi}{\partial z}$, for the flat punch is

$$V_F = \frac{F}{a} \tan^{-1} (\frac{a}{z})$$

on the z axis, where the Hertz function is

$$V_H = \frac{3F}{2a^3} ((a^2 + z^2) \tan^{-1} (a/2) - az)$$

So again $V_F = \frac{1}{3a^2} \frac{\partial}{\partial a} (a^3 V_H)$

These two related fields have simple algebraic expressions defining both the vertical displacement and applied pressure in the contact area, and were described by Boussinesq as the two simplest cases (1885 footnote p 206). He did not mention the case of the conical punch (Love (1939) Harding and Sneddon (1945) Green (1949)) which has a simple linear displacement

$$w_c = \frac{(1-\nu)F}{\pi G a^2} \left(\frac{\pi a}{2} - \rho \right)$$

in the contact circle, perhaps because of an infinity in the pressure,

$$p = \frac{F}{\pi a^2} \left(\log \left(a + (a^2 - \rho^2)^{\frac{1}{2}}\right) - \log \rho \right)$$

This conical punch field (A_3 on Table I) is also related to the flat punch solution by a simple differentiation

$$\chi_F = \frac{1}{2a} \frac{\partial}{\partial a} (a^2 \ \chi_c) \qquad (18)$$

where χ_c is the fundamental function for the conical punch. Equations similar to (18) relate corresponding components of the two fields.

The three solutions A_1, A_2 and A_3 of Table I therefore form one group, and the relations (17) and (18) between them can be used to reduce tedious calculations, or to check a derived formula.

The other three solutions B_1, B_2 and B_3 are related in a similar way. For if χ_p is the fundamental function for the parabolic pressure problem (B_2), then

$$\chi_u = \frac{1}{4a^3} \frac{\partial}{\partial a} (a^4 \chi_p)$$

is the function for Love's case of uniform pressure (B_1), and moreover

$$\chi_u = \frac{1}{3a^2} \frac{\partial}{\partial a} (a^3 \chi_L) \qquad (19)$$

where χ_L relates to the linear (or conical) pressure distribution (B_3).

The related stress fields B_1, B_2 and B_3 of this second family do not have as simple expressions for the displacement as the A group. Instead, w is expressed in terms of complete elliptic integrals of modulus (ρ/a), and the calculation of stresses is more difficult than in the cases A. However, the axial values are quite tractable, as was shown in §3, and may be found from $V(0,z)$ using equations (7). For an arbitrary point (ρ,z) the fields of B_2 and B_3 require further simplification of the integrals.

The relations (17), (18) and (19) between these different fields of elastic contact can be useful in practical calculations both as a check and as a wider view. The parabolic pressure field which we are using here can now be seen not as a strange new field but in its rightful place, being related to the uniform pressure problem [redacted] as Hertz is related to the flat punch.

5. The combined solution

This section may be read as if directly following §3.

A linear combination of the two elastic fields is formed:

$$(1 + d)\ \text{Hertz} \ - \ d\ (\text{parabolic}) \tag{20}$$

with d a positive parameter which is to be determined. Each displacement, strain or stress component of the combined field is of the form (20), with the same load F and contact radius a for the combined as for the separate fields.

Then if the combined solution is to give an indent which fits against a rigid sphere in $0 < \rho < a$, the best value of d must be found for each F, that is, the d value which makes the displaced surface closest to a sphere.

The criterion chosen for this is quite simple. Taking x, y axes not from 0 but from the lowest point of the displaced surface in fig.1, the coordinates of a surface element originally at $(\rho,0)$ are:

$$\begin{aligned} x &= \rho + u_\rho \\ y &= w_o - w \end{aligned} \tag{21}$$

where w_o is the vertical displacement at $\rho = 0$. The set of points (x,y)

corresponding to values of ρ less than a all lie on a sphere of radius R with centre on the z axis if:

$$x^2 + (R - y)^2 = R^2$$

This requires

$$(x^2 + y^2)/2y = \text{constant} = R \qquad (22)$$

and this has been used to find d for various values of the load. The fitting process determines not only d for that load but also the radius R of the sphere which would give that a at the given load F. For very small loads d is zero, since the Hertz solution is accurate there. For larger loads d increases almost linearly with the mean contact pressure (fig 4)

Values of d have been found in this way at various loads for three values of Poisson's ratio, ν = 0.1, 0.25 and 0.4. The results are shown in Table II, and also the values of the ratio a/R of the indent, with the Hertz ratio a/R_H for comparison. R differs from R_H for many of the wider contacts, but does not vary in a simple predictable way. A numerical correction factor D(d) is defined by $R = R_H/D(d)$, and this also is shown in Table II. ~~and fig 4~~.

The same results are displayed graphically in fig.3, where the straight dashed line indicates the simple proportionality of mean pressure to a/R_H as in eqn. (2), and curves are drawn for the modified relations

$$\begin{aligned} a/R &= D(d) \, . \, 3(1 - \nu)F/8\,Ga^2 \qquad (23) \\ &= D(d)\, a/R_H \end{aligned}$$

showing the effect of the correction factor D(d). This equation may also be written:

$$\begin{aligned} a^3 &= D(d) \; 3(1 - \nu)F\, R/8G \\ &= D(d) \; a^3_H \qquad (24) \end{aligned}$$

showing that $(D(d))^{1/3}$ is the correction factor for a as calculated from known F and R by the Hertz formula. The curves are drawn up to $a/R \sim 0.8$, although not many materials are capable of perfect elasticity to this extent.

It may be seen from fig.3 that for $\nu = 0.1$ the ratio a/R is always greater than the Hertz value, while for $\nu = 0.4$ it is less. For $\nu = .25$ the difference changes sign at about $a/R = 0.5$, with a/R less than the Hertz value in the upper range. The difference in the lower range is not distinguishable in the figure, but a/R exceeds the Hertz value by about 2% at $a/R \sim 0.3$.

Although $D(d) = 1$ at $a/R \sim .5$, so that $a = a_H$ and $R = R_H$, yet there are corrections to be made nevertheless. For at this point d (fig4) has the value 0.27, which modifies each component of the field according to the formula (20). For example the maximum pressure under the indenter is now given by

$$(1 + d) \frac{3F}{2\pi a^2} - d \frac{2F}{\pi a^2}$$

which is less than the Hertz value by 9%, and the depth of the indentation, from (1) and (14), is reduced by nearly 4%.

The values of d in Table II and fig.4 show that the stress field of Hertz is increasingly modified as the contact area widens, requiring larger proportions of the parabolic field to give a good fit to the sphere. But while d is increasing, the correcting factor D(d) may vary from unity in either direction. This means that, for a given series of increasing loads, a/R becomes greater than predicted by Hertz if ν is small, but less than the Hertz value for large ν, and changes over at $a/R \sim .5$ for $\nu = .25$.

The corrected values are not perfect because we are combining only two separate curves to fit a circle. After finding the best value of d, and the corresponding R, the remaining gap or overlap of sphere and surface is given by

$$[x^2 + (R - y)^2]^{\frac{1}{2}} - R$$

This remaining misfit is nearly .0003R at a/R = .5, and .001R at a/R = 0.8. Comparison with fig.2 shows the degree of improvement.

The results presented in Table II and figs. 3,4 may now be used to estimate errors in practical cases. An example is demonstrated in the next section.

6. An example

Let us suppose that the elastic modulus E of a pliant material is to be determined experimentally using a hard spherical indenter.

From the Hertz equations (1) and (2) we have, since $G = E/2(1 + \nu)$,

$$w_o = 3(1 - \nu^2)F/4Ea$$

$$= a^2/R \qquad (25)$$

and so

$$E_H = 3(1 - \nu^2)F/4R^{\frac{1}{2}} w_o^{3/2} \qquad (26)$$

where w_o is the displacement of the rigid indenter from a position just touching the original flat surface of the specimen (see Cousins, Armstrong and Robinson 1975). A series of accurate measurements of w_o and the corresponding load F, for a known indenter radius R, gives us values of $E_H/(1 - \nu^2)$, or E_H itself if ν is known.

If the values of E_H so determined appear to rise (or fall) with increasing load, then it is of interest to decide whether this is due to anelastic behaviour of the material or merely to the inaccuracy of the Hertz formulae. This question is readily answered using fig.3, which displays the calculated curves (23), and the corresponding values of d in fig.4.

Consider first the curve for $\nu = 0.1$. The factor D(d) is unity for small contact, increases to 1.05 at $a/R \sim .5$ where $d = .3$, then falls again to 1.01 at $a/R \sim .8$, $d = .55$. For w we have the combined form (20),

$$\begin{aligned} w_o &= (1 + d)\, w_o(\text{Hertz}) - d\, w_o\,(\text{parabolic}) \\ &= (1 - 0.1317d)\, 3(1 - \nu^2)F/4Ea \\ &= (1 - 0.1317d)\, a^2/RD(d) \end{aligned}$$

Substituting $a = [D(d)\, R\, w_o/(1 - 0.1317d)]^{\frac{1}{2}}$

gives an equation for E in the corrected form:

$$\begin{aligned} E &= (1 - 0.1317d)^{3/2}\, 3(1 - \nu^2)F/\left(4R^{\frac{1}{2}}\, w_o^{3/2}\, D(d)^{\frac{1}{2}}\right) \\ &= (1 - 0.1317d)^{3/2}\, E_H/D(d)^{\frac{1}{2}} \qquad (28) \end{aligned}$$

From this it appears that, for $\nu = 0.1$, the "measured" values of E_H, obtained by using eqn (26), would exceed the true values E as the load increased, the error being 9% for $a/R \sim .5$ and 12% for $a/R \sim .8$.

Repeating these calculations for $\nu = .25$ we find similar behaviour but a smaller effect, errors now 6% and 8%. Finally for $\nu = .4$ the error reaches only 1.4% at $a/R \sim .5$ then decreases to 1% at $a/R \sim .8$. The Hertz formula therefore seems reliable for E determination when ν is large, and this is fortunate, since such ν occur in some very pliant glassy polymers for which an elastic indentation may well reach high values of a/R. The error remains small because the corrections in numerator and denominator of eqn (28) are of similar magnitude in this case.

7. Conclusion

The classic formulae of Hertz, applied to the indentation of an elastic half-space by a rigid sphere, have been modified to give greater accuracy for contact areas beyond the usual range. For this purpose a large part of the elastic field of a parabolic pressure distribution has been described in detail, and also discussed more generally in relation to other contact problems.

The parabolic field has been subtracted in small proportions from the Hertz field, making use of the linearity of the system. The ratios required to give accurate fit to the sphere at wide contacts have been calculated for three different values of Poisson's ratio, and displayed graphically for practical application. Comparing the modified solution with that of Hertz, the changes in the radius of contact are in general small, with maximum 4%, but larger corrections were found for the maximum pressure and for the indentation depth.

As modern pliant materials extend the elastic range of indentation, and modern methods increase the accuracy of measurement, these results may be useful for comparison or reassurance. The theory is first order linear elasticity throughout.

Acknowledgements

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Table 1.

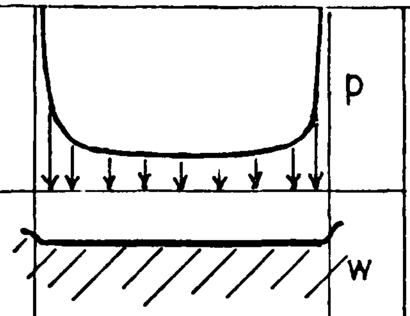
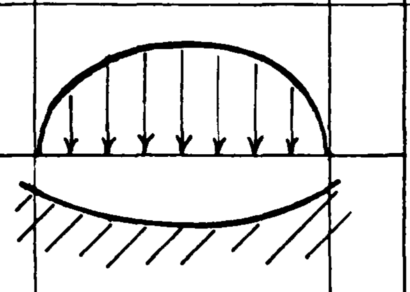
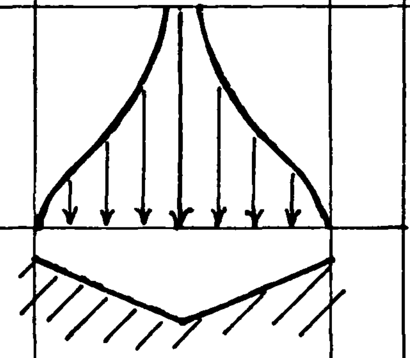
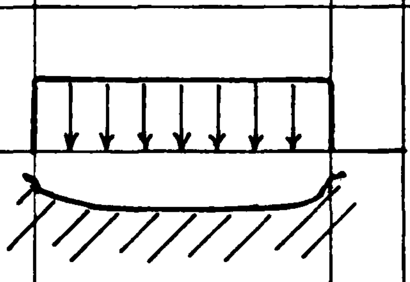
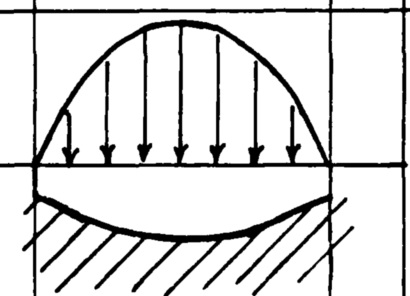
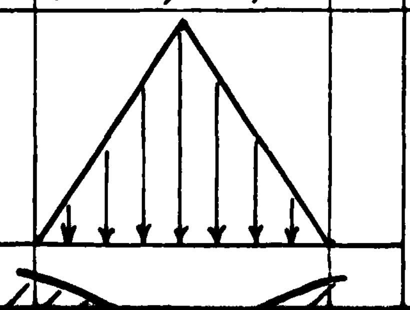
	← 2a →	Contact pressure and displacement.
A_1		Flat punch: $p = F/2\pi a(a^2-\rho^2)^{1/2}$. $w = (1-\nu)F/4Ga$.
A_2		Hertz: $p = 3F(a^2-\rho^2)^{1/2}/2\pi a^3$. $w = 3(1-\nu)F(2a^2-\rho^2)/16Ga^3$.
A_3		Cone: $p = F(\log(a+(a^2-\rho^2)^{1/2}) - \log\rho)/\pi a^2$. $w = (1-\nu)F(\pi a - 2\rho)/2\pi G a^2$.
B_1		Uniform: $p = F/\pi a^2$. $w = 2(1-\nu)F.E(k)/\pi^2 Ga$.
B_2		Parabolic: $p = 2F(a^2-\rho^2)/\pi a^4$. $w = 8(1-\nu)F(2(2-k^2E)-k'^2K)/9\pi^2 Ga$.
B_3		Linear: $p = 3F(a-\rho)/\pi a^3$. $w = 6(1-\nu)F\rho^2(\cdot 292+\int^{1} \frac{E}{k^3}\,dk)/\pi^2 Ga^3$

Table II

	$(1-\nu)F/Ga^2$	d	a/R	a/R_H	D(d)
	.45	.097	.173	.169	1.03
	.675	.15	.26	.253	1.03
ν = .1	.9	.21	.352	.337	1.04
	1.35	.32	.526	.506	1.04
	1.8	.44	.691	.675	1.02
	2.25	.58	.84	.84	1
	.187	.03	.07	.07	1
	.375	.061	.143	.14	1.02
	.562	.098	.215	.211	1.02
	.75	.14	.286	.281	1.02
ν = .25	.94	.18	.357	.352	1.02
	1.125	.22	.427	.422	1.015
	1.31	.27	.493	.491	1
	1.35	.29	.504	.506	1
	1.5	.32	.558	.562	.99
	1.875	.42	.68	.703	.97
	2.25	.52	.79	.84	.94
	.375	.04	.141	.14	1
	.6	.068	.225	.225	1
	.75	.092	.279	.281	.99
	1.2	.18	.43	.45	.96
ν = .4	1.5	.24	.53	.56	.94
	2.25	.4	.73	.84	.86

Fig.1 Diagram of indented surface showing axes and extent of contact.

Fig.2 The errors in the Hertz theory as a/R increases. Positive values of R_H - R indicate overlap, negative ones a gap.

Fig.3 Mean pressure vs modified extent of contact a/R.

Fig.4 Variation of d with mean pressure for various ν.

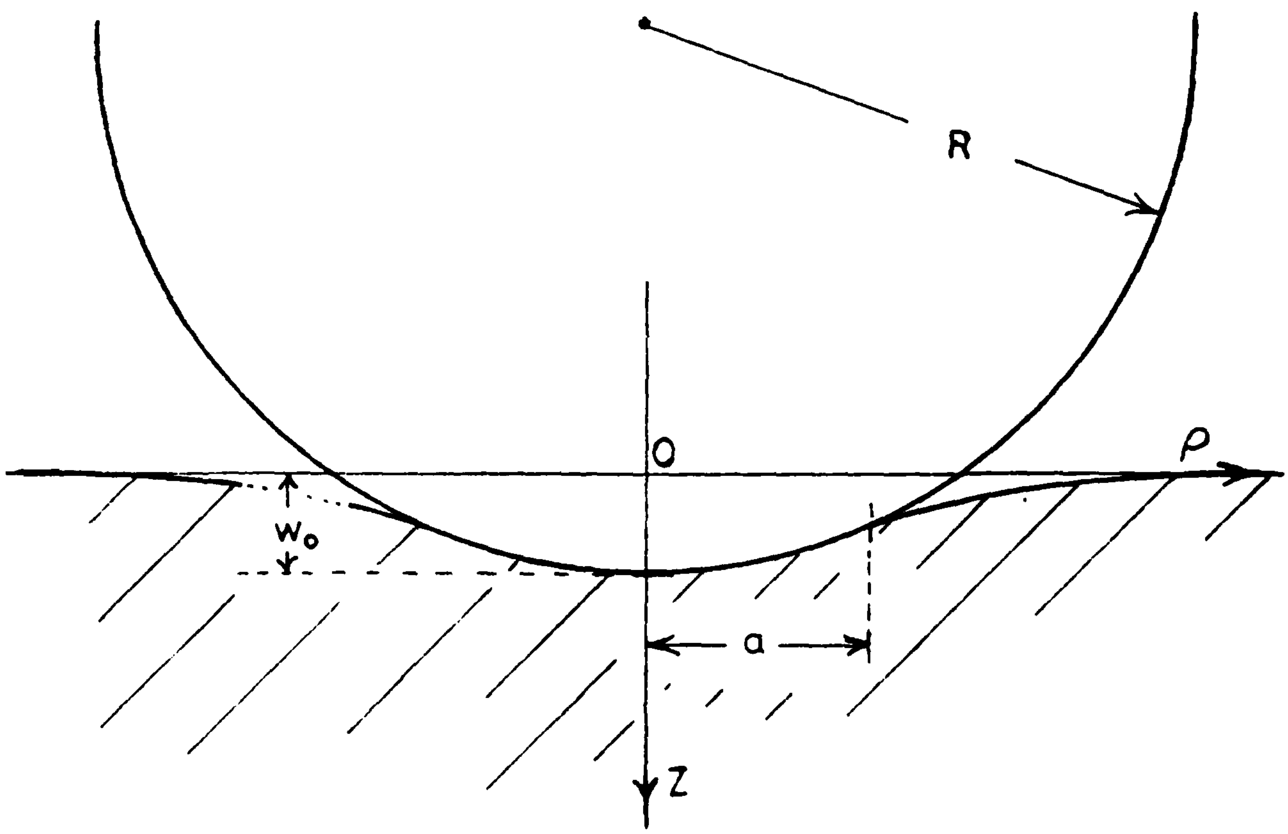


FIG.1.

Fig. 1. Diagram of indented surface showing axes and extent of contact

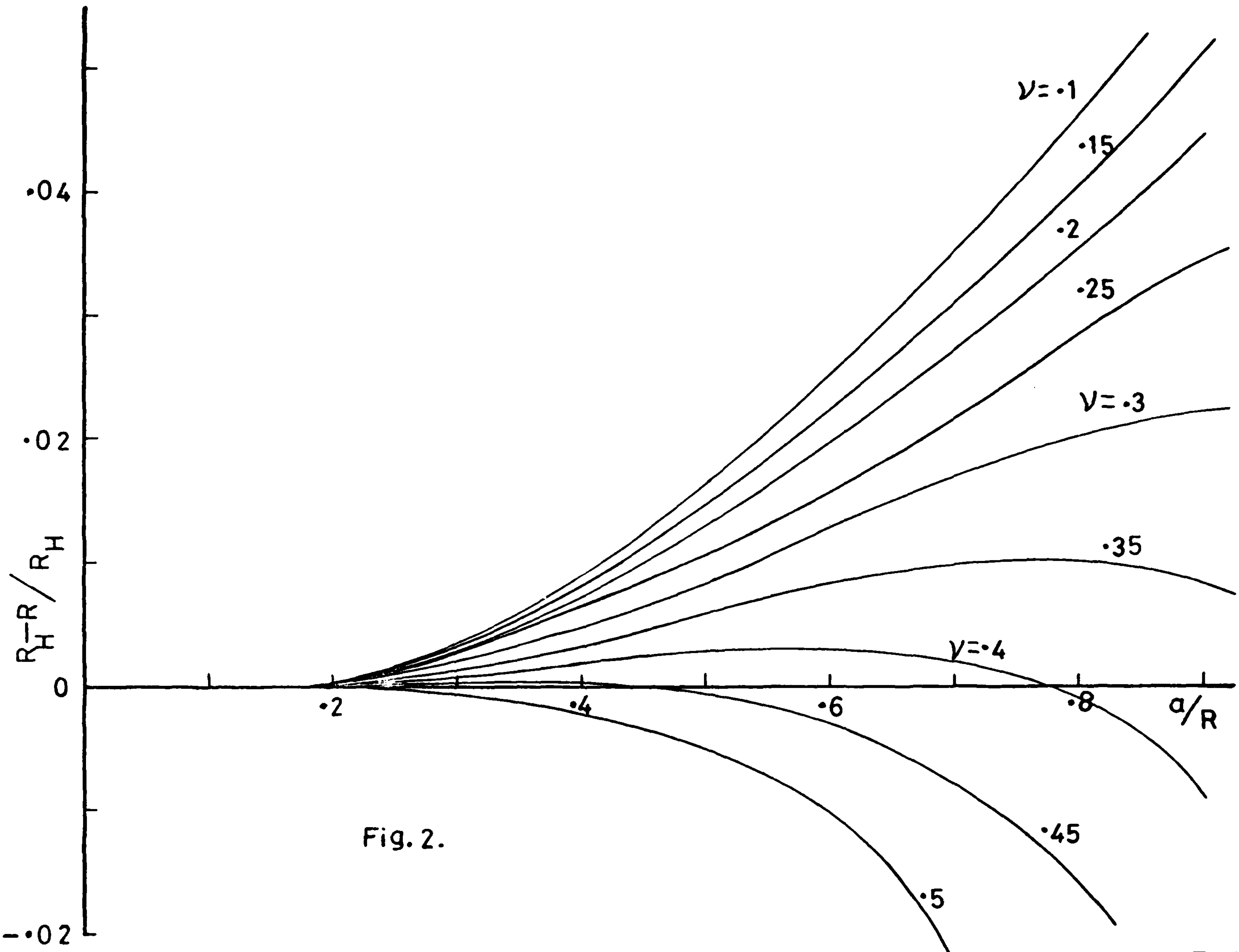
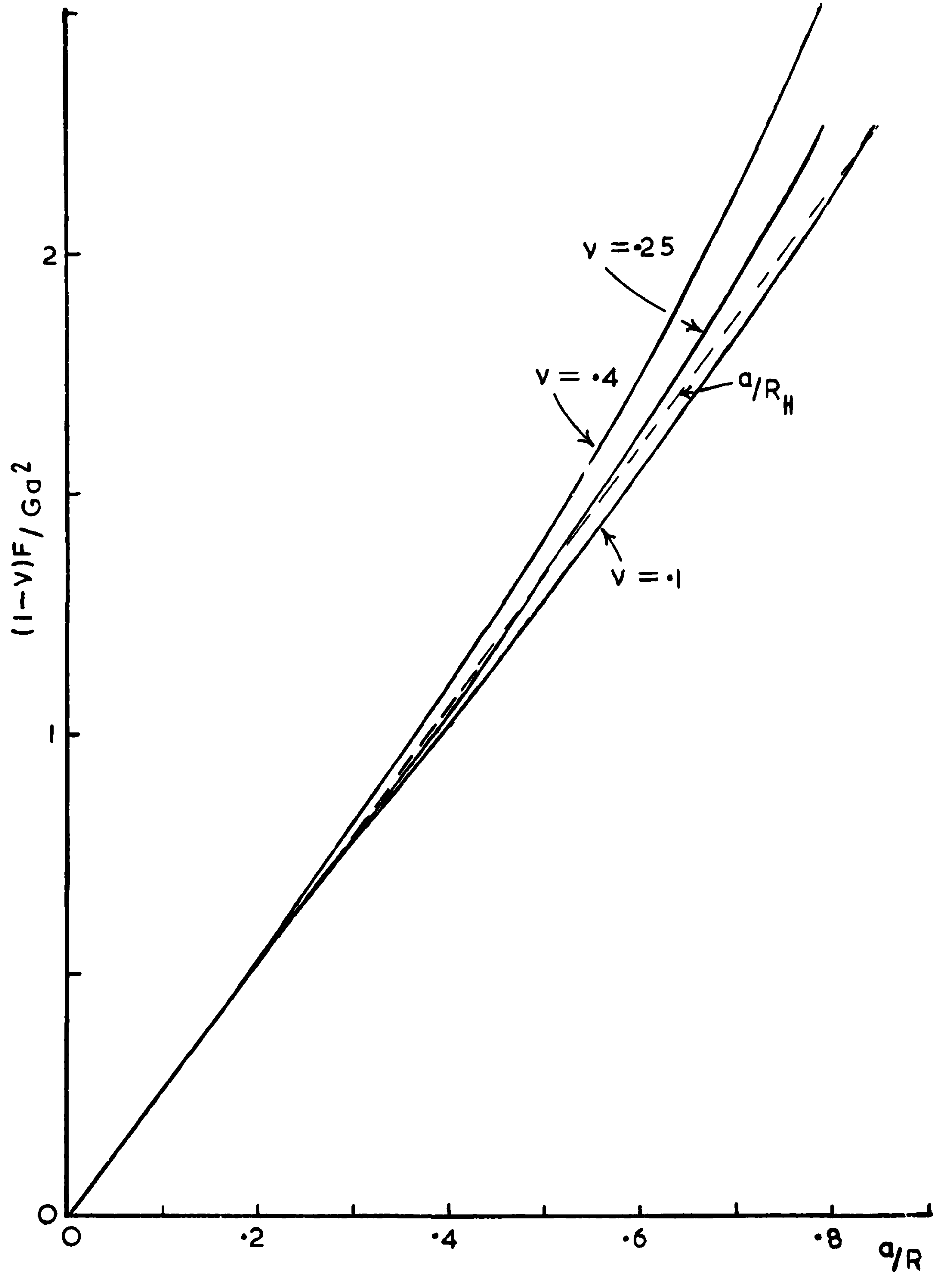
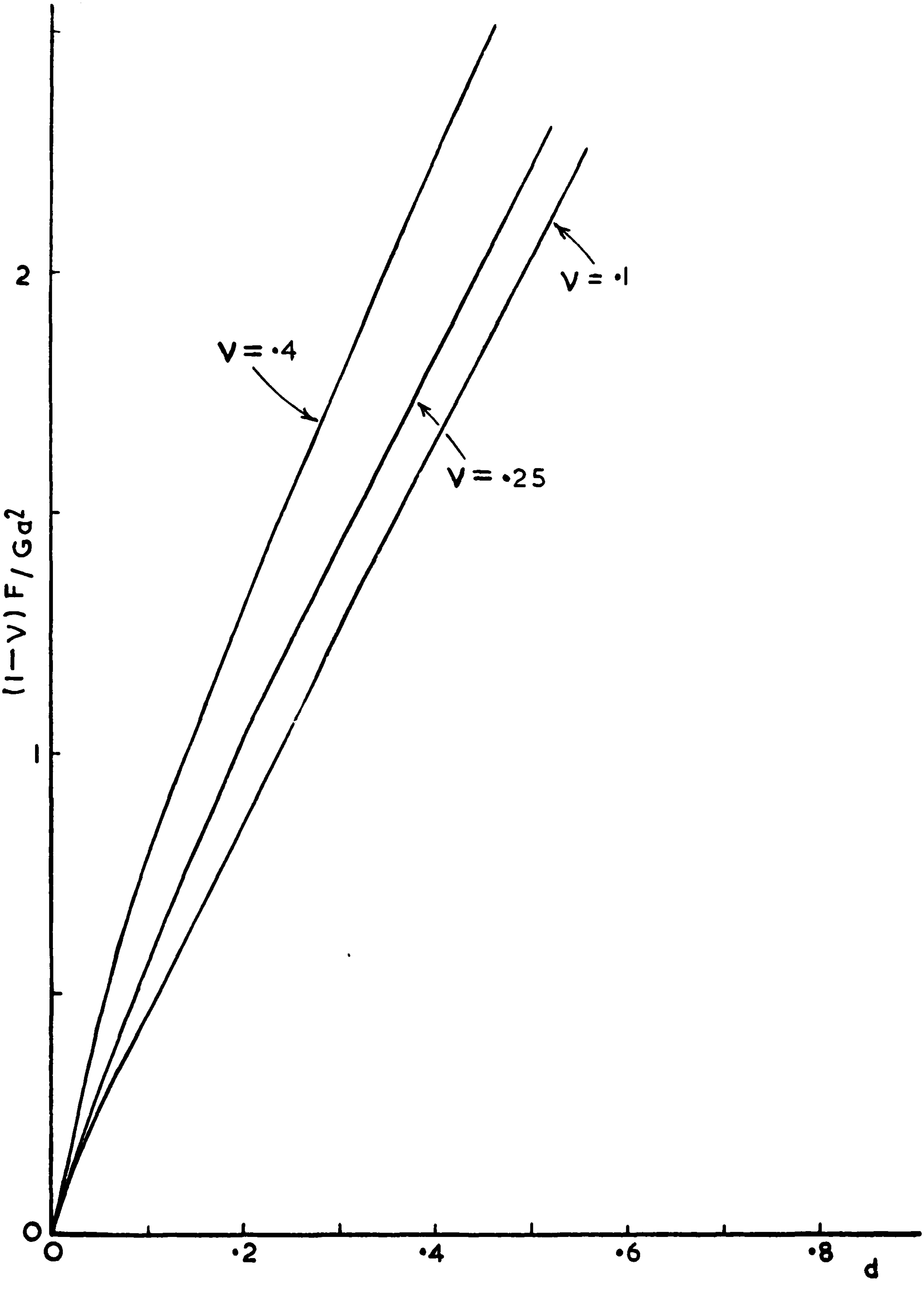


Fig. 2.



(1−ν)F / Ga²
2
1
0
O
·2
·4
·6
·8
a/R
ν = ·25
ν = ·4
a/R_H
ν = ·1



(1−V) F/Ga²
2
1
0
0
·2
·4
·6
·8
d
V = ·4
V = ·25
V = ·1

END

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